JUST-IRREGULAR p-GROUPS

BY

PAUL M. WEICHSEL*

ABSTRACT

We investigate the structure of finite irregular p-groups G, such that all proper subvarieties of var G contain only regular p-groups.

Introduction

A finite p-group G is said to be *regular* if for each $a, b \in G$ $(ab)^p = a^p b^p c^p$ where c is an element of the commutator subgroup of the group generated by a and b. This concept was first introduced by P. Hall [3] and has proved to be fundamental in the study of finite p-groups.

Regularity is inherited to subgroups and factor groups but the direct product of a pair of regular p-groups need not be regular (see e.g. [4] or [8]). This raises some interesting questions concerning varieties generated by finite p-groups (see e.g. [8], [5], [2]). In this note we consider irregular p-groups which are minimal in a very strong sense related to their varietal structure.

DEFINITION. A finite p-group G is said to be *just-irregular* if G is irregular, critical and every proper subvariety of var(G), the variety generated by G, contains no irregular groups.

The condition that G be critical is a natural one in the context of varieties and helps to eliminate some trivialities. We immediately show that this property is a refinement of the minimal condition given in [6]. Our main results are then given by

THEOREM. *Let G be a just-irregular p-group. Then*

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i) *G is a 2-generator group with exponent of G' equal to p.* [6].

ii) *G is basic.*

iii) *G has exponent p^2.*

iv) $c(G) \equiv 1 \pmod{p-1}$.

The most striking part of the theorem is that a just-irregular group must have nilpotency class either p, or $2p-1, \cdots$ or in general $1 + K(p-1)$. It is also easy to see that such a group satisfies the $(p-1)$st Engel congruence and that $(G)^p$ is the unique minimal normal subgroup of G and has order p. Thus such groups are very close to 2-generator groups of exponent p.

Notation and preliminaries

The commutator of a pair of elements a, b is denoted by $(a,b) = a^{-1}b^{-1}ab$. A left-normed commutator on $a_1, \cdots, a_n$ of weight n is defined by $(a_1, a_2, \cdots, a_n) = ((a_1, a_2, \cdots, a_{n-1}), a_n)$. A group G is said to be of nilpotency class n, denoted $c(G) = n$, if $(a_1, \cdots, a_{n+1}) = 1$ for all $a_i \in G$ but $(b_1, \cdots, b_n) \neq 1$ for some $b_1, \cdots, b_n \in G$. $\gamma_2(G) = G'$, the commutator subgroup of G, is the subgroup generated by the set $\{(a,b) \mid a, b \in G\}$ and $\gamma_n(G)$ denotes the subgroup generated by all commutators of weight n in G. $Z(G)$ denotes the center of G and $(G)^n$ is the subgroup of G generated by the set $\{a^n \mid a \in G\}$. The exponent of G, denoted $e(G)$ is the smallest positive integer k such that $x^k = 1$ for all $x \in G$. We denote by var(G) the variety generated by the group G. A group G is said to be *critical* if G is not contained in the variety generated by its proper subgroups and factor groups. G is said to be *basic* if G is critical and var(G) is not the join of proper subvarieties. Let V be a variety defined by the set of laws V. The relatively free group $F_n(V)$ of rank n in V is defined by $F_n/V(F_n)$ where F_n is the absolutely free group of rank n and $V(F_n)$ is the verbal subgroup of F_n generated by the words V. We will use the result that $F_n(V)$ contains a set $\{u_1, \cdots, u_n\}$ of elements called a free set of generators such that if $w(u_1, \cdots, u_n) = 1$, then w is a law in $F_n(V)$. The reader is referred to Hanna Neumann [7] for more details on questions concerning varieties. All groups considered are finite.

The main results

We first show that just-irregular groups are minimal in the sense defined by A. Mann [6].

LEMMA 1. *Every proper subgroup and factor group of a just-irregular group is regular.*

PROOF. Let A be a proper subgroup of the just-irregular group G. Then var(A) is a proper subvariety of var(G) and hence var(A) contains no irregular groups. Hence A is regular. Similarly if G/N is a proper factor group of G, G/N is regular.

The converse of the lemma is, of course, false as is shown by the group P in [6].

LEMMA 2. *Let G be a regular p-group satisfying $(a,b)^p = 1$ for all $a, b \in G$. Then $(ab)^p = a^p b^p$ for all $a, b \in G$.*

PROOF. This follows directly from the definition of regularity.

LEMMA 3. *If G is just-irregular, then $(G)^p \subseteq Z(G)$.*

PROOF. Let $a, b \in G$. Then the pair $\{a, b^{-1}ab\}$ generates a proper and hence regular subgroup of G. Now it follows from Lemma 1 and Theorem 2 of [6], $e(G') = p$. Thus $[(a^{-1})\ (b^{-1}ab)]^p = a^{-p}b^{-1}a^p b$ by Lemma 2. But $[(a^{-1})\ (b^{-1}ab)]^p = (a,b)^p = 1$ and $a^{-p}b^{-1}a^p b = (a^p, b)$. Hence, $(a^p, b) = 1$ and $(G)^p \subseteq Z(G)$.

LEMMA 4. *If G is just-irregular, then G is basic.*

PROOF. Every proper subvariety of var(G) is generated by regular p-groups which therefore satisfy the law $(xy)^p = x^p y^p$. If G were not basic and var(G) were a union of these, then G would also satisfy this law. Thus it would follow that G is regular, a contradiction. Hence, G is basic.

LEMMA 5. *If G is just-irregular, then $e(G) = p^2$.*

PROOF. Let $a, b \in G$. Then $(ab)^p = a^p b^p c$ with c in G'. Thus since $e(G') = p$ and $a^p, b^p \in Z(G)$ it follows that $(ab)^{p2} = a^{p2} b^{p2}$. Thus G is p^2-abelian and so by thm. 1 of [1] var(G) is the join of a subvariety generated by a p-group of exponent dividing p^2 and an abelian p-group. But G is basic and nonabelian. Hence $e(G) = p$ or p^2. But if $e(G) = p$, then G is regular. Thus, $e(G) = p^2$.

LEMMA 6. *If G is just-irregular, then $c(G) \equiv 1(\text{mod}(p-1)$.*

PROOF. Let F be the relatively free group of rank 2 in var(G), and let $\{u, v\}$ be a free set of generators of F. Since G is a 2-generator group, (see Theorem 2, [6]) var(F) = var(G). Let $c(G) = c$, and then it is clear that $F/\gamma_c(F)$ generates a proper subvariety of var(G). It follows that $F/\gamma_c(F)$ is regular and since $e(F') = p$, $F/\gamma_c(F)$ satisfies the law $(xy)^p = x^p y^p$. Hence, in F,

$$(uv)^p = u^p v^p d(u,v) \qquad (*)$$

where $d(u,v)$ is a product of powers of left-normed commutators in u and v each of weight c. Now since $\{u, v\}$ is a free set of generators for F it follows that (*) is a law for F and hence for $\mathrm{var}(F) = \mathrm{var}(G)$. Hence, we may substitute any words for u, v in (*). Let l be a primitive root of p and replace u and v by u^l and v^l respectively in (*). Thus,

$$(**) \qquad (u^l v^l)^p = u^{lp} v^{lp} d(u^l, v^l).$$

1) $d(u^l, v^l) = d(u,v)^{l^c}$. This follows from the fact that in a groups of class c, a left-normed commutator of weight c is a multilinear form of the entries and hence we can "factor out" the exponents of u^l and v^l.

2) $(u^l v^l)^p = (uv)^{lp}$. To see this consider $(uv)^l = u^l v^l w$ with $w \in G'$. Then $(uv)^{lp} = (u^l v^l w)^p = (u^l v^l)^p w^p = (u^l v^l)^p$ because the group generated by $\{u^l v^l, w\}$ is a proper subgroup of G for all choices of $u, v \in G$, and is hence regular, and so satisfies $(rs)^p = r^p s^p$.

Thus, combining (1) and (2) with (**) we get

$$(uv)^{lp} = u^{lp} v^{lp} d(u,v)^{l^c}.$$

Now raise both sides of (*) to the l-th power:

$$(uv)^{lp} = (u^p v^p d(u,v))^l = u^{lp} v^{lp} d(u,v)^l.$$

This follows because $u^p, v^p, d(u,v) \in Z(G)$.

Thus $d(u,v)^l = d(u,v)^{l^c}$. Hence, $d(u,v)^{l^c - l} = 1$. But $d(u,v) \neq 1$ for some $u, v \in G$ and so $p \mid (l^c - l) = l(l^{c-1} - 1)$. Now since l is a primitive root of p, $p \mid l^{c-1} - 1$ or $p - 1 \mid c - 1$. Thus $c - 1 = K(p-1)$ or $c \equiv 1(\mathrm{mod}(p-1))$, which was to be proven.

REMARK. Dr. A. Mann has pointed out to the author that all of the results given in the Theorem still hold if the condition that G is just-irregular is replaced by the apparently weaker condition that G is minimally irregular in the sense given in [5] and $\mathrm{var}[G]$ is join-irreducible. All that is required is a slight reworking of the proof of Lemma 6.

COROLLARY. *Let G be an irregular p-group of class $c = r + K(p-1)$ with $1 \leqq r \leqq p-1$. Then there is a chain of r subvarieties of* $\mathrm{var}(G)$

$$\mathrm{var}G = \mathbf{G}_r \supset \mathbf{G}_{r-1} \supset \cdots \supset \mathbf{G}_1$$

such that each $\mathbf{G}_i$ contains at least one irregular p-group, $i = 1, \cdots, r$.

PROOF. Since G is irregular, $\mathrm{var}G$ contains at least one just-irregular group, say H. Then $c(H) = 1 + K'(p-1)$ with $K' \leqq K$. Now let F be the relatively free group of rank 2 in var$G$. Clearly H is a factor group of $F/\gamma_{2+K'(p-1)}(F)$ and so H is contained in each of the subvarieties

$$\mathrm{var}G \supset \mathrm{var}[F/\gamma_{r+K(p-1)}(F)] \supset \cdots \supset \mathrm{var}[F/\gamma_{2+K(p-1)}(F)].$$

This is the chain required by the corollary.

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THE HEBREW UNIVERSITY OF JERUSALEM
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URBANA, ILLINOIS